Weighted Component Fairness for Forest Games

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Abstract

We present the axiom of weighted component fairness for the class of forest games, a generalization of component fairness introduced by Herings, Talman and van der Laan (2008) in order to characterize the average tree solution. Given a system of weights, component efficiency and weighted component fairness yield a unique allocation rule. We provide an analysis of the set of allocation rules generated by component efficiency and weighted component fairness. This allows us to provide a new characterization of the random tree solutions.

Key words: (Weighted) component fairness – Core – Graph games – Alexia value – Harsanyi solutions – Random tree solutions.

1 Introduction

In this paper we study TU-games with restrictions on cooperation possibilities, represented by an undirected communication graph as introduced by Myerson (1977). The links in the graph represent the bilateral communication possibilities between the agents. Agents can only cooperate if they are connected. In this paper we assume that the communication graph is represented by a forest, i.e.

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each component of the graph is a tree. A forest game is a pair consisting of a TU-game and a forest on the agent set.

Herings, van der Laan and Talman (2008) introduces a new allocation rule for the class of forest games, the so-called average tree solution. The average tree solution is the average of specific marginal contributions vectors, where each marginal contribution vector is defined according to an orientation of the links of the forest. Quite naturally, Herings, van der Laan and Talman (2008) follow Demange (2004) and envisage the set of rooted trees in each component as the set of orientations. A rooted tree is an orientation of a tree such that all links are directed away from a designated agent, called the root. Such a rooted tree describes how the agents can communicate with each other, namely: two agents cannot communicate with each other if one is not a subordinate of the other. The marginal contribution of an agent in a rooted tree is equal to the worth of the coalition consisting of this agent and all her subordinates minus the sum of the worths of the coalitions consisting of any of her successors and all subordinates of this successor. The average tree solution can be characterized by two axioms: component efficiency and component fairness. Component efficiency asserts that the members of a component ought to allocate to themselves the total worth available to them. When a link is severed in a component of a forest, this component breaks up into two new components that will be called the proper cones incident to the link. The component itself and the empty set form a cone. It turns out that this system of coalitions formed by the set of cones is a union stable system as introduced by Algaba (2001) and Bilbao (2000). Component fairness asserts that deleting a link between two agents in a component yields for both resulting proper cones the same average change in payoff, where the average is taken over the players in the component.

Recently, Béal et al. (2009) build on the work of Herings, van der Laan and Talman (2008) by relaxing the assumption that each rooted tree has the same impact in the computation of the solution. Two new sets of allocation rules are proposed for the class of tree games: the set of marginalist tree solutions and the set of random tree solutions. A marginalist tree solution is defined as a linear combination of the marginal contribution vectors and a random tree solution is defined as a probability distribution over the set of all marginal contribution vectors. The authors provided an axiomatic characterization of each of these sets of solutions.

Here, we keep component efficiency as an axiom, replace component fairness by weighted component fairness, and study the properties of the induced set of allocation rules. Combining component efficiency with component fairness implies that the members of each proper cone receives the sum of two parts. One part is the worth of the proper cone to which they belong and the second part is a share of the surplus (positive or negative) generated by the deletion of the link. This share is precisely the relative size of this proper cone. In order to explore the consequences of the change of this share on the induced set of allocations rules, we introduce a system of weights and then generalize the fairness criterion from component fairness to weighted component fairness. A system of weights assigns to each proper cone of each forest a weight between zero and one, which determines the share of the surplus between the two proper cones incident to a link.

Our first result establishes that for a fixed system of weights, component efficiency and weighted component fairness yield a unique allocation rule on the class of forest games. The expression of this rule reveals that, for each forest game, it induces a payoff vector of the cone-restricted game associated with the forest game. This cone-restricted game is defined as the restriction of the underlying TU-game on the union stable system formed by the set of cones of the forest. In addition, each agent receives the sum of two parts. The first part is the worth of the component to which he or she belongs times the payoff he or she receives in the unanimity game defined on this component. The second part is determined link by link with each of his or her neighbors. More precisely, this part of the payoff is a sum of compensation schemes between this agent and each of his or her neighbors. We provide a geometric interpretation of these compensation schemes.

Our next task is to consider the relationships between the set of allocation rules generated by component efficiency and weighted component fairness, and the core of the cone-restricted games associated with the forest games. This core is component decomposable. First, we provide a necessary and sufficient condition under which the core of a cone-restricted game is non-empty and then show that its forms a polytope in each component. Second, we establish that this core coincides with the full set of payoff vectors induced by component efficiency and weighted component fairness. In other words, a payoff vector belongs to the (non-empty) core of the cone-restricted game associated with a forest game if and only if it is obtained by component efficiency and weighted component fairness for some system of weights. Third, we show that the center of gravity of the core of a cone-restricted game coincides with the average lexicographic value of this game, also called the Alexia value, and introduced by Tijs (2005) for balanced TU-games. Because the Alexia value is a core element, it is also obtained by component efficiency and weighted component fairness for a certain system of weights. This system is such that each proper cone of a forest game has the same weight. This is equivalent to say that the Alexia value is the only component efficient rule which distributes the surplus generated by the deletion of a link equally among the two proper cones incident to this link.

Another contribution of this paper is to draw a relationship between the random tree solutions of a forest game and the Harsanyi solutions of the associated cone-restricted game. We first identify the systems of weights which generate Harsanyi payoff vectors in the corresponding cone-restricted games. We then show that among the component efficient and weighted component fair allocation rules, only the random tree solutions generate Harsanyi payoff vectors in the associated cone-restricted games. This allows us to provide an alternative characterization of the random tree solutions in terms of component efficiency and weighted component fairness.

As final way of calculating the allocation rules introduced in this paper, we highlight their combinatorial nature by underlying that all these rules are equal to a weighted average of the solutions of Cramer systems constructed from a subset of the orientations of the forest and the TU-game associated with a forest game. For instance, each random tree solution can be viewed as a weighted average of these solutions, where the average is taken from the set of all rooted trees. In the say way, the Alexia payoff vector is the average of the these solutions over the full set of orientations.

This paper is organized as follows. Section 2 contains definitions and a preliminary result. The axiom of weighted component fairness is introduced in section 3. We establish that, for a given system of weights, component efficiency and weighted component fairness yield a unique allocation rule on the class of forest games. In section 4, we investigate the structure of the core of the cone-restricted game. Section 5 is devoted to the relationship between the Harsanyi solutions of the cone-restricted game and the random tree solutions.

2 Preliminaries

2.1 Cooperative games

Consider a finite set of agents $N = \{1, 2, \ldots, n\}$. Let 2^N be the set of all subsets of N partially ordered by the set inclusion $\subseteq$. Any subset of 2^N inherits the set inclusion order from 2^N. A *coalition* is an element S of 2^N, and N is called the *grand coalition*. For a coalition S, the small letter s denotes its cardinal. In many practical situations some coalitions may not be meaningful. This may due to the lack of communication possibilities, or certain institutional constraints. It means that some coalitions are not feasible, so that the partially ordered set is no more 2^N, but a subcollection Ω of it. Several models of restricted cooperation have been proposed. We refer the reader to the book of Bilbao (2000). Here, we will light upon the union stable systems as introduced by Algaba et al (2001). A subset of coalitions Ω is *union stable* if for any two feasible coalitions $S, T \in \Omega$ such that $S \cap T \neq \emptyset$, then $S \cup T \in \Omega$. This condition means that if two feasible coalitions have common elements, these ones will act as intermediaries between the two coalitions in order to establish meaningful cooperation in the whole group.

A *cooperative game* with transferable utility is a pair (N, v) where N is the set of agents and $v : \Omega \longrightarrow \mathbb{R}$ is the *characteristic function* on the set of feasible coalitions Ω. It is assumed that $\emptyset \in \Omega$ and $v(\emptyset) = 0$. The characteristic function v assigns to every coalition $S \in \Omega$ its *worth* $v(S)$, which is interpreted as the maximal value the members of S can obtain by agreeing to cooperate. The set Ω_0 denotes the set of all non-empty coalitions in Ω.

By $\mathcal{C}(\Omega)$, we denote the real linear space of all characteristic functions v on Ω, which can be identified with $\mathbb{R}^{\omega-1}$, where ω stands for the size of Ω. Each $v \in \mathcal{C}(\Omega)$ can be expressed in a unique way as:

$$v = \sum_{T \in \Omega} a_v(T) u_T,$$

where the real numbers $a_v(T)$, $T \in \Omega_0$, are called the *Harsanyi dividends* of v and where the collection of games $\{u_T : \Omega \longrightarrow \mathbb{R} : T \in \Omega_0\}$, called the *unanimity games*, are given by: $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise.

A *payoff vector* $x \in \mathbb{R}^n$ of $v \in \mathcal{C}(\Omega)$ is a vector giving a payoff $x_i \in \mathbb{R}$ to any agent $i \in N$. For a non-empty coalition $S \in \Omega$ and a payoff vector x, the notation x_S stands for $\sum_{i \in S} x_i$.

A *solution* on $\mathcal{C}(\Omega)$ is a mapping F that assigns to every $v \in \mathcal{C}(\Omega)$ a set of payoff vectors $F(v) \in \mathbb{R}^n$. In case $F(v)$ is single-valued, we say that F is an *allocation rule*, and we will use the small letter f instead of the capital letter F.

In this paper, we consider the class of *Harsanyi solutions* proposed by Vasil'ev (1982) and applied recently by van den Brink et al. (2007) to graph-restricted games (see Section Solutions for graph games). First a *sharing system* on Ω_0 is a system $p = (p^S)_{S \in \Omega_0}$, where p^S is an s-dimensional vector assigning to each player $i \in N$ a share p_i^S such that:

$$p_i^S \in [0,1] \text{ and } \sum_{i \in S} p_i^S = 1.$$

For a sharing system p, the corresponding Harsanyi solution f^p induces the Harsanyi payoffs defined as:

$$\forall v \in \mathcal{C}(\Omega), \forall i \in N, \quad f_i^p(v) = \sum_{\substack{S \in \Omega_0: \\ i \in S}} p_i^S a_v(S).$$

2.2 Graph games

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents who face restrictions on communication possibilities. The bilateral communication possibilities between the agents are represented by an *undirected graph* on N, denoted by (N, L), where the set of nodes coincides with the set of agents N, and the set of links L is a subset of the set of unordered pairs $\{i, j\}$ of elements of N. For simplicity, we write ij to represent the link $\{i, j\}$, and so $ij \in L$ indicates that i and j are linked in (N, L). The notation L_{-ij} stands for the set of links obtained by deleting the link ij from the set of links L.

For each agent $i \in N$, the set $L_i = \{j \in N : ij \in L\}$ denotes the *neighborhood* of i in (N, L). The *degree* of an agent $i \in N$ in (N, L), denoted by d_i, is the number of elements of L_i. For each non-empty coalition S of N, $L(S) = \{ij \in L : i \in S, j \in S\}$ is the set of links between agents in S. Note that $L(N) = L$. The graph $(S, L(S))$ is the *subgraph* of (N, L) induced by S. A sequence of distinct agents $(i_1, i_2, \ldots, i_p)$ is a *path* in (N, L) if $i_k i_{k+1} \in L$ for $k = 1, \ldots, p-1$. Two agents i and j are *connected* in (N, L) if $i = j$ or there exists a path from i to j. A graph (N, L) is *connected* if any two agents in N are connected. A *tree* is a minimally connected graph (N, L) in the sense that only one path connects any two agents. Note that a tree on N has exactly $n-1$ links. A coalition S is connected in (N, L) if $(S, L(S))$ is a connected graph. The empty coalition $\emptyset$ is trivially connected. Note that the set of connected coalitions of a graph is union stable. A coalition C of N is a *component* of a graph (N, L) if the subgraph $(C, L(C))$ is maximally connected, i.e. if the subgraph $(C, L(C))$ is connected and for each $i \in N \backslash C$, the subgraph $(C \cup \{i\}, L(C \cup \{i\}))$ is not connected. Note that the collection of components of (N, L) forms a partition of N. Let N/L be the collection of components of a graph (N, L). The concept of component is defined similarly for each subgraph $(S, L(S))$. A *forest* is a graph (N, L) such that each subgraph $(C, L(C))$ induced by a component $C \in N/L$ is a tree, i.e a forest is a collection of disconnected trees. Note that a forest with c components and n nodes has $n - c$ links.

The set of $cones$ of a forest (N, L) consists of the set of components N/L, the set $\emptyset$, and for each $i, j \in C \in N/L$ such that $ij \in L$ the two components of the subgraph of $(C, L(C))$ that are obtained after the deletion of the link ij. Every cone strictly included in C is a $proper\ cone$ of C. The unique agent of a non-empty proper cone K who has a link with the complement of $C \backslash K$ is called the $head$ of K and is denoted by $h(K)$. Observe that $K \subseteq C$ is a cone if and only if $C \backslash K$ is a connected coalition. If K is a cone, then the cone $C \backslash K$ is the complement of K in C and is denoted by K^c. In order to insist on the link ij, $K_{(j,i)}$ denotes the cone with head i and $K_{(i,j)}$ its complement with head j in the corresponding component. Denote by Δ_L and by Δ_L^0 the set of cones and the set of non-empty proper cones of a forest (N, L) respectively. The set Δ_L is union stable since the union of two cones with a non-empty intersection is the component to which they belong or one of the cones in case they are comparable with respect to set inclusion.

An $orientation$ $\circ$ of a graph (N, L) is a $directed\ graph$ $(N, L_\circ)$ obtained from (N, L) by replacing each link ij by either the directed link (i, j) or the directed link (j, i). Each component $C \in N/L$ of a forest (N, L) admits exactly 2^{c-1} orientations. Denote by $\circ(ij)$ the orientation of a link $ij \in L$.

A $graph\ game$ on N is a pair (v, L) such that (N, v) is a cooperative game in $\mathcal{C}(2^N)$ and (N, L) is a communication graph on N. In what follows, we consider only the set of $forest\ games$ on N, denoted by $\mathcal{F}$. As for cooperative games on $\mathcal{C}(\Omega)$, an allocation rule on $\mathcal{F}$ is a map f on $\mathcal{F}$ which assigns to every forest game (v, L) an n-dimensional payoff vector $f(v, L)$.

2.3 Solutions for graph games

While for the class $\mathcal{C}(\Omega)$ it is explicitly assumed that only coalitions in Ω are feasible, a graph game (v, L) indicates that the entire set of coalitions 2^N may be affected by (N, L). The payoff vector $f(v, L)$ determines precisely how the communication links affect both the formation of coalitions and the process of redistribution. Therefore, $f(v, L)$ can be expressed as a payoff vector $g(w)$ on some class $\mathcal{C}(\Omega)$ where w and Ω incorporate both the possible gains from cooperation as modeled by v and the restrictions on communication reflected by the communication graph. Myerson (1977) is the first to study graph games. He introduces the so-called *graph-restricted game* v^L on $\mathcal{C}(2^N)$ associated with a graph game (v, L). The characteristic function v^L is defined as follows:

$$\forall S \in 2^N, \quad v^L(S) = \sum_{C \in S/L(S)} v(C).$$

This means that if S is a connected coalition, then its members can cooperate and obtain the worth $v(S)$. Otherwise, not all agents in S can communicate with each other, and the coalition is partitioned into components according to $S/L(S)$. The best that the members of S can accomplish under these communication constraints is to cooperate within each component. Myerson (1977) introduces the Shapley value (Shapley, 1953) of $v^L \in \mathcal{C}(2^N)$, also known as the Myerson value for graph games, i.e the Myerson value $\mathsf{M}(v, L) = \mathsf{Sh}(v^L)$ where $\mathsf{Sh}(\cdot)$ denotes the Shapley value. The Shapley value is the Harsanyi solution associated with the sharing system p given by $p_i^S = 1/s$. Van den Brink et al. (2007) discuss Harsanyi solutions for v^L and provide an axiomatic characterization of these solutions.

The Myerson value can be characterized by two properties: component efficiency and fairness. Component efficiency requires that the payoffs in a component add up to the worth of that component.

Component efficiency. A solution f on $\mathcal{F}$ is *component efficient* if for each $(v, L) \in \mathcal{F}$ and each $C \in N/L$, it holds that:

$$\sum_{i \in C} f_i(v, L) = v(C).$$

Fairness requires that the payoffs of two agents incident to the same link increase or decrease by the same amount when the link is severed.

Fairness. A solution f on $\mathcal{F}$ is *fair* if for each $(v, L) \in \mathcal{F}$ and each $ij \in L$ it holds that:

$$f_i(v, L) - f_i(v, L_{-ij}) = f_j(v, L) - f_j(v, L_{-ij}).$$

Herings, van der Laan and Talman (2008) discuss the axiom of fairness and suggest to replace it by the alternative axiom of component fairness. This axiom says that deleting a link between two agents yields for both resulting cones the same average change in payoffs. Component fairness therefore emphasizes that in forest games the gains associated to linking one cone to its complement in a component should be attributed to these two proper cones, rather than to the two agents whose link is deleted, and the gains should be proportional to the the size of the cones.

Component fairness. A solution f on $\mathcal{F}$ satisfies *component fairness* if for each $(v, L) \in \mathcal{F}$ and each link $ij \in L$, it holds that:

$$\frac{1}{k_{(j,i)}}\bigg(f_{K_{(j,i)}}(v, L) - f_{K_{(j,i)}}(v, L_{-ij})\bigg) = \frac{1}{k_{(i,j)}}\bigg(f_{K_{(i,j)}}(v, L) - f_{K_{(i,j)}}(v, L_{-ij})\bigg) \tag{2.1}$$

Herings, van der Laan and Talman (2008) show that component efficiency and component fairness characterize a new allocation rule called the Average Tree solution. The Average Tree solution is a marginalist rule in the sense that it can be expressed as the average of marginal contribution vectors. Each of these vectors calibrates the importance of each agent in the different coalitions by taking into account the communication possibilities. To describe these vectors some definitions concerning rooted trees are in order. By a *rooted tree* t_r^C on the subgraph $(C, L(C))$, we mean an orientation that arises from a component C of a forest (N, L) by selecting agent $r \in C$, called the *root*, and directing all links of $L(C)$ away from the root r. Because r belongs to exactly one component of (N, L), we will use the notation t_r instead of t_r^C when no confusion arises. Each agent $r \in N$ is the root of exactly one rooted tree t_r. Note also that for any rooted tree t_r on (N, L), any agent $k \in C\backslash\{r\}$, there is exactly one directed link (j, k); agent j is the unique *predecessor* of k and k is a *successor* of j in t_r. Denote by $s_r(j)$ the possibly empty set of successors of agent j in t_r. An agent k is a *subordinate* of j in t_r if there is a directed path from j to k, i.e. if there is a sequence of distinct agents $(i_1, i_2, \ldots, i_p)$ such that $i_1 = j$, $i_p = k$ and for each $q = 1, 2, \ldots, p-1$, $i_{q+1} \in s_r(i_q)$. The set $S_r(j)$ denotes the union of the set of all subordinates of j in t_r and $\{j\}$. So, we have $s_r(j) \subseteq S_r(j)\backslash\{j\}$. A rooted tree reflects the idea that two agents incident to a communication link do not have equal access or control to that link.

Pick any $(v, L) \in \mathcal{F}$, any component $C \in N/L$, any root $r \in C$, and consider the marginal contribution vector $m^r(v, L)$ on $\mathbb{R}^c$ defined as:

$$\forall i \in C, \quad m_i^r(v, L) = v(S_r(i)) - \sum_{j \in s_r(i)} v(S_r(j)) \tag{2.2}$$

The marginal contribution $m_i^r(v, L)$ of $i \in C$ in t_r is thus equal to the worth of the coalition consisting of agent i and all his subordinates in t_r minus the sum of the worths of the coalitions consisting of any successor of i and all subordinates of this successor in t_r.

The *Average Tree solution* is the allocation rule AT on $\mathcal{F}$ which assigns to each $(v, L) \in \mathcal{F}$ the payoff vector in which agent i in a component C receives the average over $r \in C$ of the payoffs $m_i^r(v, L)$:

$$\forall C \in N/L, \forall i \in C, \quad \mathsf{AT}_i(v, L) = \frac{1}{c} \sum_{r \in C} m_i^r(v, L) \tag{2.3}$$

For further developments on the average tree solution, see Herings, van der Laan and Talman (2008) and Baron et al. (2009). Béal et al. (2009) introduce and characterize the set of Random Tree solutions on the class of forest games. An allocation rule on $\mathcal{F}$ is a *Random Tree solution*, denoted by RT^q, if for each forest (N, L) and each component $C \in N/L$, there is a probability distribution $q^C = (q^C(r))_{r \in C}$ over the set of c rooted trees in $(C, L(C))$ such that:

$$\forall C \in N/L, \forall i \in C, \quad \mathsf{RT}_i^q(v, L) = \sum_{r \in C} q^C(r) m_i^r(v, L).$$

Thus, AT is the Random Tree solution where $q^C(r) = 1/c$.

Two remarks motivate this paper. First, note that in (2.1), the weight of each proper cone $K_{(j,i)} \subset C$, $ij \in L(C)$, can be measured by its relative size $k_{(j,i)}/c$ since (2.1) is equivalent to the following expression:

$$\frac{k_{(i,j)}}{c}\Big(f_{K_{(j,i)}}(v, L) - f_{K_{(j,i)}}(v, L_{-ij})\Big) = \frac{k_{(j,i)}}{c}\Big(f_{K_{(i,j)}}(v, L) - f_{K_{(i,j)}}(v, L_{-ij})\Big).$$

Our aim is to extend the axiom of component fairness by considering all the possible weights for the proper cones. The corresponding axiom will be called weighted component fairness. Second, as noted by Béal et al. (2009), only cones of (N, L) are used to compute a Random Tree solution $\mathsf{RT}^q(v, L)$. Therefore, for each forest (N, L), the map $v \longmapsto \mathsf{RT}^q(v, L)$ determines an allocation rule f on the set of *cone-restricted games* $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$, where each v^{Δ_L} is defined as the restriction of v to the union stable system $\Delta_L \subset 2^N$, i.e. for all $K \in \Delta_L$, $v^{\Delta_L}(K) = v(K)$. In other words, we have $\mathsf{RT}^q(v, L) = f(v^{\Delta_L})$. When considering all the possible weights for the proper cones, the induced solutions are also payoff vectors of the cone-restricted game. We will show that these payoff vectors have some interesting properties.

Before introducing this new axiom, we present a preliminary result which will be very useful for the rest of the paper. Let (N, L) be a forest. Pick any component $C \in N/L$ and consider the directed subgraph $(C, L_\circ(C))$ obtained from the orientation $\circ$. Define a collection of real numbers $\{b^K : K \in \Delta_L\}$ such that:

$$\forall K \in \Delta_L^0, \quad b^C = b^K + b^{K^c} \tag{2.4}$$

From $(C, L_\circ(C))$ and $\{b^K : K \in \Delta_L\}$, construct the following system of c linear equations with c unknowns:

$$\sum_{k \in C} x_k = b^C \text{ and } \sum_{k \in K_{(j,i)}} x_k = b^{K_{(j,i)}} \text{ for each directed link } (j,i) \in L_\circ(C), \tag{2.5}$$

where $K_{(j,i)}$ denotes the proper cone whose head is i and obtained by deleting the directed link (j,i). Because the directed subgraph $(C, L_\circ(C))$ has exactly $c-1$ directed links, this procedure selects $c-1$ proper cones plus the cone C.

For each forest game $(v, L) \in \mathcal{F}$ and each directed subgraph $(C, L_\circ(C))$, $C \in N/L$, obtained from the orientation $\circ$, we will consider the linear system of the form (2.5) with constant terms $b^C = v(C)$ and for each directed link (j,i), $b^{K_{(j,i)}} = v(K_{(j,i)})$. Such a system will be called a $\circ$-*system associated with* (v, L) *in component* C. Denote by $x^\circ(v, L)$ a solution of such a linear system.

Lemma 2.1 *For each collection $\{b^K : K \in \Delta_L\}$ and each orientation $\circ$ of $(C, L(C))$, the system (2.5) admits exactly one solution.*

Proof. Consider any collection $\{b^K : K \in \Delta_L\}$ satisfying (2.4), and any orientation $\circ$ of $(C, L(C))$. Pick any directed link $(j,i) \in L_o(C)$. If we substitute the equation $\sum_{k \in K_{(j,i)}} x_k = b^{K_{(j,i)}}$ in (2.5) by the complementary equation $\sum_{k \in K_{(i,j)}} x_k = b^C - b^{K_{(j,i)}}$, then we get an equivalent system. Thus, it suffices to prove that a system of c linear equations including the equation $\sum_{k \in C} x_k = b^C$ and, for each $(j,i) \in L_o$, a linear equation either of the form:

$$\sum_{k \in K_{(j,i)}} x_k = b^{K_{(j,i)}} \text{ or } \sum_{k \in K_{(i,j)}} x_k = b^C - b^{K_{(j,i)}}$$

has a unique solution. To construct such a system, we consider an *ordering* of the elements of C, i.e. a bijective function σ on C. Given an ordering σ on C, $\sigma(i)$ is the agent at position $i \in C$ in this ordering. We define the ordering σ such that $\sigma(1)$ is a leaf of the tree $(C, L(C))$ and for each $i \in C$, $\sigma(i)$ is a leaf of the (sub)tree obtained by deleting the set $\{\sigma(1), \sigma(2), \ldots, \sigma(i-1)\}$. Let $\{\sigma(i), \sigma(j)\} \in L(C)$ and assume without loss of generality that $i < j$. We have the following facts: $\sigma(i)$ is a leaf of the tree obtained by deleting the set $\{\sigma(1), \sigma(2), \ldots, \sigma(i-1)\}$; $\sigma(i)$ is linked to $\sigma(j)$ in this tree; for each $k \in C$ such that $\sigma(k) \in K_{(\sigma(j),\sigma(i))}$, it holds that $k \leq i$. By construction, for each $i \in C$ there is a unique cone $K_{\sigma(i)}$ such that $\sigma(i)$ is the head of $K_{\sigma(i)}$ and $k \leq i$ for each $\sigma(k) \in K_{\sigma(i)}$. In particular, $K_{\sigma(c)} = C$. Now, order the unknowns according to σ and choose the equation associated with the cone $K_{\sigma(i)}$ for each $\sigma(i)$. We get a linear system of the following form:

$$\forall i \in C, \qquad \sum_{\sigma(k) \in K_{\sigma(i)}} x_{\sigma(k)} = b^{K_{\sigma(i)}}.$$

The above system is lower triangular and each diagonal term is equal to 1. Thus, it admits exactly one solution. $\square$

Note that each marginal contribution vector $m^r(v, L) \in \mathbb{R}^c$ as defined in (2.2) is the unique solution of the t_r-system associated with (v, L) in component C, where t_r is the orientation inducing the directed tree t_r rooted at $r \in C$. Hence, a Random Tree payoff vector $(\mathsf{RT}_i^q(v, L))_{i \in C}$ is a convex combination of these solutions over the c orientations that induce the c rooted trees of $(C, L(C))$.

3 Weighted component fairness

In this section, we introduce the axiom of weighted component fairness. A *system of weights* on the set of forests on N is a function α which assigns to each L the system of weights

$$\alpha(L) = \big\{\alpha_K(L) : K \in \Delta_L^0, \alpha_K(L) \in [0,1], \alpha_K(L) + \alpha_{K^c}(L) = 1\big\}.$$

Let $\mathcal{A}$ be the set of all systems of weights α.

α-component fairness. Given a system of weights $\alpha \in \mathcal{A}$, an allocation rule f on $\mathcal{F}$ satisfies *α-component fairness* if for each (v, L) and each link $ij \in L$, it holds that:

$$\alpha_{K_{(i,j)}}(L)\big(f_{K_{(j,i)}}(v, L) - f_{K_{(j,i)}}(v, L_{-ij})\big) = \alpha_{K_{(j,i)}}(L)\big(f_{K_{(i,j)}}(v, L) - f_{K_{(i,j)}}(v, L_{-ij})\big).$$

The axiom of component fairness as introduced in (2.1) corresponds to the special case where $\alpha_{K_{(i,j)}}(L) = k_{(i,j)}/c$, where c is the size of the component C and $k_{(i,j)}$ is the size of the proper cone $K_{(i,j)} \subset C$.

Theorem 3.1 *For each system of weights $\alpha \in \mathcal{A}$ on the set of forests on N, there is a unique allocation rule f^{α} on $\mathcal{F}$ that satisfies component efficiency and α-component fairness. Moreover, for each $(v, L) \in \mathcal{C}_N$ it holds that $f^{\alpha}(v, L) = g^{\alpha}(v^{\Delta_L})$ where for each $C \in N/L$ and each $i \in C$,*

$$g_i^{\alpha}(v^{\Delta_L}) = v(C) - \sum_{j \in L_i} v(K_{(i,j)}) - \sum_{j \in L_i} \alpha_{K_{(i,j)}}(L)\big(v(C) - v(K_{(j,i)}) - v(K_{(i,j)})\big) \tag{3.1}$$

Proof. Suppose that an allocation rule f satisfies component efficient and α-component fairness for some $\alpha \in \mathcal{A}$. Pick any $(v, L) \in \mathcal{C}_N$ and any component $C \in N/L$ of size c. Component efficiency implies that

$$f_C(v, L) = v(C) \tag{3.2}$$

By component efficiency, we also have for each $ij \in L(C)$, $f_{K_{(i,j)}}(v, L_{-ij}) = v(K_{(i,j)})$, $f_{K_{(j,i)}}(v, L_{-ij}) = v(K_{(j,i)})$, and $f_{K_{(j,i)}}(v, L) = v(C) - f_{K_{(i,j)}}(v, L)$. Therefore, using the fact that $\alpha_{K_{(i,j)}}(L) + \alpha_{K_{(j,i)}}(L) = 1$, α-component fairness reduces to

$$f_{K_{(i,j)}}(v, L) = v(K_{(i,j)}) + \alpha_{K_{(i,j)}}(L)\big(v(C) - v(K_{(j,i)}) - v(K_{(i,j)})\big) \tag{3.3}$$

For each component C of size c, there is one linear equation (3.2) and $c-1$ linear equations of type (3.3). It follows that equations (3.2) and (3.3) constitute a linear system of c linear equations with c unknowns. This linear system is of the form of the system (2.5) where the constant terms are: $b^C = v(C)$, and

$$\forall ij \in L(C),\ b^{K_{(i,j)}} = v(K_{(i,j)}) + \alpha_{K_{(i,j)}}(L)\big(v(C) - v(K_{(j,i)}) - v(K_{(i,j)})\big).$$

Clearly, these constant terms satisfy (2.4). By Lemma 2.1, this linear system has a unique solution $(f_i^{\alpha}(v, L))_{i \in C}$. Continuing in this fashion for each component C of (N, L), we conclude that component efficiency and α-component fairness yield a unique solution $f^{\alpha}(v, L)$. Next, pick any $i \in C$. By component efficiency, we have:

$$f_i^{\alpha}(v, L) = v(C) - \sum_{j \in L_i} f^{\alpha}_{K_{(i,j)}}(v, L).$$

Using the expression of $f^{\alpha}_{K_{(i,j)}}(v, L)$ given in (3.3), we obtain:

$$f_i^{\alpha}(v, L) = v(C) - \sum_{j \in L_i} v(K_{(i,j)}) - \sum_{j \in L_i} \alpha_{K_{(i,j)}}(L)\big(v(C) - v(K_{(i,j)}) - v(K_{(j,i)})\big).$$

The right-hand side of this expression is precisely the right-hand side of (3.1). Since only cones of (N, L) are used to compute $f^{\alpha}(v, L)$, the result follows. $\square$

Expression (3.1) of $g_i^\alpha(v^{\Delta_L})$ will prove very useful in the next section. Nevertheless, as such it is not very appealing. The payoff $g_i^\alpha(v^{\Delta_L})$ can be rewritten is such a way that it is equal to the sum of two parts: the first part is a multiple of the payoff obtained in the unanimity game $u_C^{\Delta_L}$ and the second part is determined for each link $ij \in L$ by the area of the parallelogram formed by the vectors $(\alpha_{K_{(i,j)}}, 1-\alpha_{K_{(i,j)}})$ and $(v(K_{(i,j)}, v(K_{(j,i)}))$, which represent the parallelogram's sides. To see this, consider again expression (3.1):

$$\begin{aligned} g_i^\alpha(v^{\Delta_L}) &= v(C) - \sum_{j\in L_i} v(K_{(i,j)}) - \sum_{j\in L_i} \alpha_{K_{(i,j)}}(L)\big(v(C) - v(K_{(j,i)}) - v(K_{(i,j)})\big) \\ &= v(C)\left(1 - \sum_{j\in L_i} \alpha_{K_{(i,j)}}(L)\right) \\ &\quad + \sum_{j\in L_i}\left(\alpha_{K_{(i,j)}}(L)v(K_{(j,i)}) - \big(1-\alpha_{K_{(i,j)}}(L)\big)v(K_{(i,j)})\right) \\ &= v(C)g_i^\alpha(u_C^{\Delta_L}) + \sum_{j\in L_i} \det A_{ij}^{v,\alpha} \end{aligned} \tag{3.4}$$

where $\det A_{ij}^{v,\alpha}$ denotes the determinant of the 2×2 matrix $A_{ij}^{v,\alpha}$ defined as:

$$A_{ij}^{v,\alpha} = \begin{pmatrix} v(K_{(j,i)}) & \alpha_{K_{(j,i)}}(L) \\ v(K_{(i,j)}) & \alpha_{K_{(i,j)}}(L) \end{pmatrix}.$$

The payoff $g_i^\alpha(u_C^{\Delta_L})$ is directly obtained from (3.1). The determinant $\det A_{ij}^{v,\alpha}$ can be viewed as the oriented area of the parallelogram with vertices at (0,0), $(v(K_{(j,i)}), v(K_{(i,j)}))$, $(\alpha_{K_{(j,i)}}(L), \alpha_{K_{(i,j)}}(L))$ and $(v(K_{(j,i)}) + \alpha_{K_{(j,i)}}(L), v(K_{(i,j)}) + \alpha_{K_{(i,j)}}(L))$. The oriented area is the same as the usual area, except that it is negative when the vertices are listed in clockwise order. Note also that the allocation process associated with a link ij has the zero-sum property: agent i receives $\det A_{ij}^{v,\alpha}$ and agent j receives exactly $-\det A_{ij}^{v,\alpha}$. In order to interpret this situation, assume that the connected component C forms. Because $L(C)$ is minimally connected all communication links are necessary to coordinate the actions inside C. Suppose that the link $ij \in L(C)$ is broken. How should i and j be compensated? The determinant $\det A_{ij}^{v,\alpha}$ offers a compensation scheme between i and j whose geometric interpretation in the form of an orientated area is quite natural.

Lemma 2.1 and Theorem 3.1 are also useful to construct new allocations rules. By equation (3.3) in Theorem 3.1, each allocation rule satisfying component efficiency and α-component fairness distributes to the members of the cone $K_{(i,j)}$ the sum of two parts. One part is the worth of $K_{(i,j)}$ and the second part is a share $\alpha_{K_{(i,j)}}(L)$ of the surplus $v(C)-(v(K_{(j,i)})+v(K_{(i,j)}))$ generated by the creation/deletion of the link ij. For instance, the allocation rule AT distributes this surplus according to the relative size of the cone. Instead, consider the case where this surplus is equally distributed to the members of the two cones incident to the link ij. The associated system of weights, denoted by e, is such that $e_K(L)=1/2$ for each forest (N,L) and each proper cone $K\in\Delta_L^0$. From this, we derive the following axiom of e-component fairness.

e-component fairness. An allocation rule f on $\mathcal{F}$ satisfies *e-component fairness* if for each $(v,L)\in\mathcal{F}$ and each link $ij\in L$, it holds that:

$$f_{K_{(j,i)}}(v,L)-f_{K_{(j,i)}}(v,L_{-ij})=f_{K_{(i,j)}}(v,L)-f_{K_{(i,j)}}(v,L_{-ij}) \tag{3.5}$$

The unique allocation rule f^e satisfying component efficiency and e-component fairness will be called the *Egalitarian Cone-Surplus Sharing Rule*. As for AT, the allocation rule f^e is equal to the average of the solutions of a set of linear systems of the form (2.5). More precisely, we have the following result.

Theorem 3.2 *For each forest game $(v,L)\in\mathcal{F}$, each component $C\in N/L$ and each agent $i\in C$, the payoff $f_i^e(v,L)$ is equal to the average over the set of all orientations of the solutions of the $\circ$-systems associated with (v,L) in component C.*

Proof. Pick any $(v,L)\in\mathcal{F}$, any $C\in N/L$. From (3.3), we have:

$$\forall ij\in L(C),\quad f^e_{K_{(j,i)}}(v,L)=v(K_{(j,i)})+\frac{v(C)-v(K_{(j,i)})-v(K_{(i,j)})}{2}.$$

On the other hand, consider the solutions $x^\circ(v,L)=(x_k^\circ(v,L))_{k\in C}$ of the $\circ$-systems associated with (v,L) in component C. There are exactly 2^{c-1} such linear systems, one for each orientation $\circ$ of the subgraph $(C,L(C))$. Define the payoff vector $(\mathsf{AO}(v,L))_{k\in C}$ as the average of the solutions $x^\circ(v,L)$ over the set of all orientations:

$$\forall k\in C,\quad \mathsf{AO}_k(v,L)=\frac{1}{2^{c-1}}\sum_{\circ}x_k^\circ(v,L).$$

Proceeding in this way for each component $C\in N/L$, we obtain a unique n-dimensional payoff vector $\mathsf{AO}(v,L)$.

Pick any link $ij \in L(C)$ and compute the aggregate payoff vector $\mathsf{AO}_{K_i}(v, L)$. We get:

$$\mathsf{AO}_{K_{(i,j)}}(v,L) = \frac{1}{2^{c-1}}\Bigg[\sum_{\circ:\circ(ij)=(j,i)} v(K_{(j,i)}) + \sum_{\circ:\circ(ij)=(i,j)} \big(v(C) - v(K_{(i,j)})\big)\Bigg].$$

Obviously, there are as much orientations such that $\circ(ij) = (j, i)$ as orientations such that $\circ(ij) = (i, j)$. Thus, we have:

$$\begin{aligned}\mathsf{AO}_{K_{(j,i)}}(v,L) &= \frac{1}{2^{c-1}}\Big[2^{c-2}v(K_{(j,i)}) + 2^{c-2}\big(v(C) - v(K_{(i,j)})\big)\Big] \\ &= f^e_{K_{(j,i)}}(v,L).\end{aligned}$$

In the same way, we obtain $\mathsf{AO}_{K_{(j,i)}}(v, L) = f^e_{K_{(j,i)}}(v, L)$. By component efficiency of f^e, we get: $\mathsf{AO}_{K_{(i,j)}}(v, L) + \mathsf{AO}_{K_{(j,i)}}(v, L) = v(C)$. Since this equality holds for each $(v, L) \in \mathcal{F}$, each $C \in N/L$ and each link $ij \in L(C)$, we have shown that AO satisfies component efficiency. By component efficiency, we also obtain:

$$\begin{aligned}\mathsf{AO}_{K_{(j,i)}}(v,L) &= v(K_{(j,i)}) + \frac{v(C) - v(K_{(j,i)}) - v(K_{(j,i)})}{2} \\ &= \mathsf{AO}_{K_{(j,i)}}(v,L_{-ij}) + \frac{v(C) - \mathsf{AO}_{K_{(j,i)}}(v,L_{-ij}) - \mathsf{AO}_{K_{(i,j)}}(v,L_{-ij})}{2},\end{aligned}$$

and so

$$\frac{1}{2}\bigg(\mathsf{AO}_{K_{(j,i)}}(v,L) - \mathsf{AO}_{K_{(j,i)}}(v,L_{-ij})\bigg) = \frac{1}{2}\bigg(\mathsf{AO}_{K_{(i,j)}}(v,L) - \mathsf{AO}_{K_{(i,j)}}(v,L_{-ij})\bigg),$$

which is precisely the axiom of e-component fairness expressed in (3.5). Therefore, AO satisfies component efficiency and e-component fairness on $\mathcal{F}$. By Theorem 3.1, we conclude that $f^e = \mathsf{AO}$, as desired. □

In view of Theorem 3.2, the only difference between $\mathsf{AT}(v, L)$ and $f^e(v, L)$ is the subset of orientations from which the average of the solutions $x^\circ(v, L)$ is taken. Besides the fact that f^e coincides with the Average Orientation solution AO on $\mathcal{F}$, it possesses interesting features that will be studied in the next section. In particular, it is worth noting that $x^\circ(x, L)$ can equivalently be obtained on $\mathcal{F}$ by component efficiency and α-component fairness for a system of weights $\alpha \in \mathcal{A}$ such that $\alpha_{K_{(j,i)}}(L) = 0$ if $(j, i) \in L_\circ(C)$ and $\alpha_{K_{(j,i)}}(L) = 1$ otherwise.

4 Fairness, the core and the Alexia value

Fix a forest (N, L). Theorem 3.1 indicates that for each system of weights α, the map $v \longmapsto f^{\alpha}(v, L)$ determines an allocation rule $g^{\alpha}(v^{\Delta_L})$ on $\mathcal{C}(\Delta_L)$. In this section, we study the properties of these solutions on $\mathcal{C}(\Delta_L)$.

A payoff vector $x \in \mathbb{R}^n$ is *acceptable* with respect to the union stable system Δ_L if $x_K \geq v(K)$ for each $K \in \Delta_L$. The *core* is the most well-known component efficient solution for cooperative games. Given a cone-restricted game v^{Δ_L}, the core of v^{Δ_L} is the convex set of payoff vectors that are both acceptable and component efficient, i.e. it is the possibly empty set defined as:

$$\text{Core}(v^{\Delta_L}) = \Big\{ x \in \mathbb{R}^n : \forall C \in N/L, x_C = v(C), \text{ and } \forall K \in \Delta_L^0, x_K \geq v(K) \Big\}.$$

A payoff vector $x \in \mathbb{R}^n$ is an *extreme point* of the core if there do not exist distinct payoff vectors z and y in Core(v^{Δ_L}) and $a \in]0, 1[$ such that $x = az + (1 - a)y$.

Assume that we narrow attention to the (non-empty) core of a cone-restricted game. If we want only one of all these payoff vectors how can we do it? Tijs (2005)[1] suggests an allocation rule for cooperative games with a non-empty core, called the average lexicographic value or the Alexia value. The Alexia value is defined as the average of the leximals, where a leximal is defined as a lexicographical maximum of the core, with respect to an arbitrary ordering on the agents. In a leximal, the amount allocated to an agent is the maximum he or she can obtain within the core, under the constraint that the agents before him or her in the corresponding ordering recursively obtain their restricted maximum.

Let Σ_N the set of $n!$ orderings σ on N. Given an ordering σ on N, $\sigma(i)$ is the agent at position $i \in N$ in this ordering For a game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$ with a non-empty core, the *leximal* $\lambda^{\sigma}(v^{\Delta_L}) \in \mathbb{R}^n$ is the payoff vector defined as the lexicographic maximum with respect to the ordering σ, i.e. for each $i \in N$,

$$\lambda^{\sigma}_{\sigma(i)}\big(v^{\Delta_L}\big) = \max \Big\{ x_{\sigma(i)} \in \mathbb{R} : x \in \text{Core}\big(v^{\Delta_L}\big), \lambda^{\sigma}_{\sigma(k)}\big(v^{\Delta_L}\big) = x_{\sigma(k)}, k < i \Big\}.$$

For a game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$ with a non-empty core, the *Alexia value* of v^{Δ_L},

[1] Tijs (2005) introduces the Alexia value for the class of games with a non-empty core belonging to $\mathcal{C}(2^N)$. Here, we adapt the definition to the class of cooperative games with a non-empty core in $\mathcal{C}(\Delta_L)$.

denoted by $\mathsf{AL}\big(v^{\Delta_L}\big)$, is defined as the average of the leximals, i.e.

$$\mathsf{AL}\big(v^{\Delta_L}\big) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} \lambda^{\sigma}(v^{\Delta_L}).$$

A cooperative game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$ is $cone\text{-}modular$ if for each link $ij \in L$, it holds that:

$$v(C) \geq v(K_{(j,i)}) + v(K_{(i,j)}),$$

where C denotes the unique component containing i and j.

The first result of this section establishes that a cooperative game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$ has a non-empty core if and only if it is cone-modular. Moreover, the core of a cone-modular game v^{Δ_L} coincides with the set of all allocations $g^{\alpha}(v^{\Delta_L})$ obtained by component efficiency and α-component fairness on $\mathcal{F}$. In this sense, the core of a cone modular game is fair.

Theorem 4.1 *A game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$ has a non-empty core if and only if it is cone-modular. Moreover, for each cone-modular game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$ we have:*

$$\textit{Core}\big(v^{\Delta_L}\big) = \Big\{ x \in \mathbb{R}^n : x = g^{\alpha}(v^{\Delta_L}), \alpha \in \mathcal{A} \Big\}.$$

Proof. Pick any game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$. It is obvious that $\mathsf{Core}\big(v^{\Delta_L}\big) = \emptyset$ if v^{Δ_L} is not cone-modular. So, assume that v^{Δ_L} is cone-modular. Since the core is component decomposable, it is sufficient to focus the analysis on the case where (N, L) has a single component. So, assume without loss of generality that N is the unique component of (N, L). We proceed in four steps.

(a) We show that $\mathsf{Core}\big(v^{\Delta_L}\big)$ is a polytope. The core is a convex polyhedron, so it remains to verify that it constitutes a bounded set. Consider any $i \in N$ and any payoff vector $x \in \mathsf{Core}\big(v^{\Delta_L}\big)$. To show: there exist $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ such that $a_i \leq x_i \leq b_i$. Recall that d_i denotes the number of elements of L_i, $i \in N$. For each $j \in L_i$, we have: $x_{K_{ji}} \geq v(K_{ji})$. Summing over the neighbors of agent i, we get:

$$\sum_{j \in L_i} x_{K_{ji}} \geq \sum_{j \in L_i} v(K_{ji}), \text{ or equivalently, } d_i x_i \geq \sum_{j \in L_i} v(K_{ji}) - \sum_{j \in L_i} x_{K_{ji} \setminus \{i\}}.$$

Because any agent in $N \setminus \{i\}$ belongs to all but one of the cones with head i, we can rewrite the previous inequality as follows:

$$d_i x_i \geq \sum_{j \in L_i} v(K_{ji}) - (d_i - 1) x_{N \setminus \{i\}}.$$

Finally, by efficiency of x we obtain the following lower bound for x_i:

$$x_i \geq \sum_{j \in L_i} v(K_{ji}) - (d_i - 1)v(N). \tag{4.1}$$

The upper bound for x_i is obtained by replacing in the equation of efficiency the payoff x_k of any agent $k \in N \backslash \{i\}$ by (4.1). Conclude that Core$\big(v^{\Delta_L}\big)$ is a polytope and thus is defined as the convex hull of its extreme points.

(b) We determine the profile of Core$\big(v^{\Delta_L}\big)$, i.e. the collection of all extreme points of Core$\big(v^{\Delta_L}\big)$. We claim that the profile of Core$\big(v^{\Delta_L}\big)$ is given by:

$$\Big\{ x \in \mathbb{R}^n : x = x^\circ(v, L) \text{ for some orientation} \circ \Big\} \tag{4.2}$$

Pick any orientation $\circ$ of (N, L). The solution $x^\circ(v, L) \in \mathbb{R}^n$ of the $\circ$-system associated with (v, L) in component N belongs to Core$\big(v^{\Delta_L}\big)$. To see this, note that by construction, the solution $x^\circ(v, L)$ is component efficient, and for each $(j, i) \in L_o$, it holds that $x^\circ_{K_{(j,i)}}(v, L) = v(K_{(j,i)})$, and $x^\circ_{K_{(i,j)}}(v, L) = v(N) - v(K_{(j,i)}) \geq v(K_{(i,j)})$ by component efficiency of $x^\circ(v, L)$ and cone-modularity of v^{Δ_L}.

Assume, by way of contradiction, that $x^\circ(v, L)$ is not an extreme point of Core(v^{Δ_L}). Then, we can choose x and y in Core(v^{Δ_L}) such that $x \neq x^\circ(v, L)$ and $x^\circ(v, L) = (x + y)/2$. There necessarily exists a cone $K_{(j,i)}$ for which $x_{K_{(j,i)}} \neq x^\circ_{K_{(j,i)}}(v, L)$. One can assume without loss of generality that $x_{K_{(j,i)}} < x^\circ_{K_{(j,i)}}(v, L)$. If (j, i) is an element of $L_\circ$, then $x^\circ_{K_{(j,i)}}(v, L) = v(K_{(j,i)})$, which contradicts the fact that $x \in$ Core(v^{Δ_L}). Thus, (i, j) is necessarily an element of $L_\circ$ and $x^\circ_{K_{(i,j)}}(v, L) = v(K_{(i,j)})$. By component efficiency, we have $x_{K_{(i,j)}} > x^\circ_{K_{(i,j)}}(v, L) = v(K_{(i,j)})$. It follows that $y_{K_{(i,j)}} < x^\circ_{K_{(i,j)}}(v, L) = v(K_{(i,j)})$, which contradicts the fact that $y \in$ Core(v^{Δ_L}). Therefore, the solution $x^\circ(v, L)$ is an extreme point of Core(v^{Δ_L}).

To complete the proof of the claim, let x be an extreme point of Core(v^{Δ_L}). We show that at least one of the following equalities holds for each link $ij \in L$: $x_{K_{(j,i)}} = v(K_{(j,i)})$ or $x_{K_{(i,j)}} = v(K_{(i,j)})$. Suppose this claim is false, i.e. suppose there exists at least one link $ij \in L$ such that both equalities are violated. Consider any such link $ij \in L$, choose any $\epsilon \in \mathbb{R}$ and construct the payoff vector x^ϵ as follows: $x^\epsilon_i = x_i + \epsilon$, $x^\epsilon_j = x_j - \epsilon$ and $x^\epsilon_k = x_k$ for each $k \in N \setminus \{i, j\}$. For $|\epsilon|$ sufficiently small, it easy to see that x^ϵ and $x^{-\epsilon}$ belong to Core(v^{Δ_L}) and $x = (x^\epsilon + x^{-\epsilon})/2$, which contradicts the premise that x is an extreme point of

Core(v^{Δ_L}). Thus the claim is true. It immediately follows that we can orientate all links in such a way that for each directed link (j,i) of L_o, $x_{K_{(j,i)}} = v(K_{(j,i)})$. By definition and Lemma 2.1, $x = x^o(v,L)$. Therefore, the profile of Core(v^{Δ_L}) coincides with the set given in (4.2), as claimed.

(c) We claim that:

$$\text{Core}\big(v^{\Delta_L}\big) \subseteq \Big\{x \in \mathbb{R}^n : x = g^{\alpha}(v^{\Delta_L}), \alpha \in \mathcal{A}\Big\}.$$

For each pair α^1 and α^2 in $\mathcal{A}$, and each $a \in [0,1]$, we have $a\alpha^1 + (1-a)\alpha^2 \in \mathcal{A}$ and

$$ag^{\alpha^1}(v^{\Delta_L}) + (1-a)g^{\alpha^2}(v^{\Delta_L}) = g^{a\alpha^1+(1-a)\alpha^2)}(v^{\Delta_L}),$$

where the equality follows from equation (3.1) in Theorem 3.1. This means that

$$\Big\{x \in \mathbb{R}^n : x = g^{\alpha}(v^{\Delta_L}), \alpha \in \mathcal{A}\Big\} \tag{4.3}$$

is a convex set of payoff vectors. On the other hand, we see that each element $x^{\circ}(v,L)$ belonging to the profile of Core$\big(v^{\Delta_L}\big)$ is obtained on $\mathcal{F}$ by component efficiency and α-component fairness, where $\alpha \in \mathcal{A}$ is such that $\alpha_{K_{(j,i)}}(L) = 0$ if $(j,i) \in L_{\circ}(C)$ and $\alpha_{K_{(ji)}}(L) = 1$ otherwise. So, $x^{\circ}(v,L) = g^{\alpha}(v^{\Delta_L})$ for such a system of weights $\alpha \in \mathcal{A}$. Since Core$\big(v^{\Delta_L}\big)$ is a convex hull of its profile by point (a), the claim follows by convexity of the set (4.3).

(d) It remains to show the reverse inclusion:

$$\Big\{x \in \mathbb{R}^n : x = g^{\alpha}(v^{\Delta_L}), \alpha \in \mathcal{A}\Big\} \subseteq \text{Core}\big(v^{\Delta_L}\big).$$

Pick any payoff vector $g^{\alpha}(v^{\Delta_L})$. By equation (3.3) in Theorem 3.1, we know that:

$$\forall K \in \Delta_L^0, \quad g_K^{\alpha}(v^{\Delta_L}) = v(K) + \alpha_K(L)\big(v(N) - v(K^c) - v(K)\big).$$

By cone-modularity of v^{Δ_L}, $v(N) - v(K^c) - v(K) \geq 0$, and so $g_K^{\alpha}(v^{\Delta_L}) \geq v(K)$, which means that $g^{\alpha}(v^{\Delta_L})$ is acceptable for all proper cones of (N,L). By component efficiency of $g^{\alpha}(v^{\Delta_L})$, we conclude that $g^{\alpha}(v^{\Delta_L})$ is acceptable for all cones of (N,L), which means that $g^{\alpha}(v^{\Delta_L}) \in$ Core$\big(v^{\Delta_L}\big)$, as desired. □

In case $\Omega = 2^N$, Shapley (1971) shows that the Shapley value is the center of gravity of the core – the average of its extreme points – of each supermodular game belonging to $\mathcal{C}(2^N)$. Tijs (2005) shows that the Alexia value coincides with the Shapley value in supermodular games. The next theorem extends this result on the class of games $\mathcal{C}(\Delta_L)$. More precisely, we establish that the Egalitarian Cone-Surplus Sharing Rule selects the Alexia value in each cone-modular game. In other words, e-component fairness and component efficiency on $\mathcal{F}$ selects the Alexia value in each cone-modular v^{Δ_L} associated with the forest game (v, L), and the Alexia value in the center of gravity of this cone-modular game.

Theorem 4.2 *The payoff vector $g^e(v^{\Delta_L})$ of a cone-modular game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$ is the center of gravity of Core$\big(v^{\Delta_L}\big)$ and $g^e(v^{\Delta_L}) = AL\big(v^{\Delta_L}\big)$.*

Proof. See supplementary material. □

5 Harsanyi solutions

In this section, we show that an allocation rule g^α, $\alpha \in \mathcal{A}$, on $\mathcal{C}(\Delta_L)$ is a Harsanyi solution if and only if it is a Random Tree solution. We also provide a new characterization of the Random Tree solutions on $\mathcal{F}$ in terms of component efficiency and α-component fairness. Let $\mathcal{H}$ be the set of Harsanyi solutions on $\mathcal{C}(\Delta_L)$. Note that $\mathcal{H}$ is a convex set.

Theorem 5.1 *Let g^α, $\alpha \in \mathcal{A}$, be an allocation rule on $\mathcal{C}(\Delta_L)$. The following assertions are equivalent.*

1. *The allocation rule g^α belongs to $\mathcal{H}$.*

2. *The system of weights α is such that for each $i \in N$, it holds that*

$$\sum_{j \in L_i} \alpha_{K_{(i,j)}}(L) \leq 1 \tag{5.1}$$

3. *The allocation rule g^α is a Random Tree solution, i.e. $g^\alpha(v^{\Delta_L}) = RT^q(v, L)$ for each cone-restricted game $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$.*

Notation: in the following proof, u_K stands for $u_K^{\Delta_L}$ where K is a non-empty of (N, L).

Proof. (1) $\iff$ (2). Pick any $\alpha \in \mathcal{A}$ and define the system $p = (p^K)_{K \in \Delta_L \setminus \{\emptyset\}}$ as follows: for each non-empty cone $K \in \Delta_L$, p^K is a k-dimensional real vector assigning to each $i \in K$ the value

$$p_i^K = 1 - \sum_{j \in L_i \cap K} \alpha_{K_{(i,j)}}(L) \tag{5.2}$$

Note that the system p depends only on the forest (N, L).

By (3.1), g^α is a linear allocation rule on $\mathcal{C}(\Delta_L)$ so that for each $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$, we have:

$$g^\alpha(v^{\Delta_L}) = \sum_{K \in \Delta_L \setminus \{\emptyset\}} a_{v^{\Delta_L}}(K) g^\alpha(u_K).$$

Consider a non-empty cone $K \subseteq C$, $C \in N/L$. Observe that for each $i \in C \setminus K$ and each link $ij \in L(C)$, either $K_{(i,j)} \supseteq K$ or $K_{(j,i)} \supseteq K$, and there is exactly one link $ij \in L(C)$ such that $K_{(i,j)} \supseteq K$. From this observation and from (3.1) in Theorem 3.1, we immediately get:

$$g_i^\alpha(u_K) = \begin{cases} p_i^K & \text{if } i \in K \\ 0 & \text{if } i \in N \setminus K. \end{cases}$$

In case $C = K$, we see from the expression of p^C that every agent $i \in C$ gets precisely the payoff expressed in (5.1):

$$g_i^\alpha(u_C) = 1 - \sum_{j \in L_i} \alpha_{K_{(i,j)}}(L).$$

Therefore, we can rewrite $g^\alpha(v^{\Delta_L})$ as follows:

$$\forall i \in N, \quad g_i^\alpha(v^{\Delta_L}) = \sum_{\substack{K \in \Delta_L: \\ i \in K}} p_i^K a_{v^{\Delta_L}}(K).$$

By component efficiency of g^α, we obtain

$$\sum_{i \in C} g_i^\alpha(u_K) = \sum_{i \in K} g_i^\alpha(u_K) = \sum_{i \in K} p_i^K = 1.$$

Therefore, the system p is a sharing system if and only if for each non-empty cone K and each $i \in K$, $p_i^K \in [0, 1]$, i.e. if and only if (5.1) holds. We conclude that g^α is a Harsanyi solution on $\mathcal{C}(\Delta_L)$ if and only if (5.1) holds.

(1) $\iff$ (3). To show: for any $v^{\Delta_L} \in \mathcal{C}(\Delta_L)$, $g^{\alpha}(v^{\Delta_L})$ is a Harsanyi payoff vector if and only if $g^{\alpha}(v^{\Delta_L})$ is a Random Tree payoff vector.

The set of payoff vectors $g^{\alpha}(v^{\Delta_L})$, $\alpha \in \mathcal{A}$, is a convex set by Theorem 4.1. The profile of this convex set, given by (4.2), is generated by the following systems of weights. For L and each component $C \in N/L$, there is an orientation $\circ$ of $(C, L(C))$ such that for each orientated link (j, i), $\alpha_{K_{(i,j)}}(L) = 1$, and so $\alpha_{K_{(j,i)}}(L) = 0$. Using component efficiency and α-component fairness on $\mathcal{F}$, the induced payoff vectors $g^{\alpha}(v^{\Delta_L})$ are such that $g^{\alpha}_i(v^{\Delta_L}) = x^{\circ}_i(v, L)$, $i \in C$, $C \in N/L$. If an orientation $\circ$ is not a rooted tree, there is a least one agent who has at least two predecessors in the induced orientated subgraph and so (5.1) is violated. In case $\circ$ is a rooted tree, each agent belonging to this rooted tree has at most one predecessor and (5.1) is satisfied. Thus, the systems of weights constructed in this manner satisfy (5.1) on L if and only if the orientations $\circ$ on the components of L are rooted trees. In such a case, by (3.1) in Theorem 3.1, $g^{\alpha}_i(v^{\Delta_L}) = m^r_i(v, L)$, $i \in C$, $C \in N/L$. By the preceding point, we know that assertion 2 is equivalent to assertion 1. Therefore, we conclude that among the elements of the profile (4.2), only the marginal contribution vectors are Harsanyi payoffs vectors, where the sharing system p is constructed as in (5.2) from the vector of weights $\alpha(L)$ defined above. Because a convex combination of Harsanyi payoff vectors is still a Harsanyi payoff vector, the set of Random tree payoff vectors – the convex hull of the marginal vectors – belongs to the set of Harsanyi payoff vectors.

It remains to show that the set of Harsanyi payoff vectors of the form $g^{\alpha}(v^{\Delta_L})$, $\alpha \in \mathcal{A}$, belongs to the set of Random tree payoff vectors. It suffices to show that the marginal contribution vectors constitutes the profile of the convex set of these Harsanyi payoff vectors. Suppose, by way of contradiction, that this assertion is false. Thus, there exists a Harsanyi payoff vector $g^{\alpha}(v^{\Delta_L})$ in this profile such that for at least one link $ij \in L$ we have $\alpha_{K_i}(L)$ and $\alpha_{K_j}(L)$ in $]0, 1[$. Let $(i_1, i_2, \ldots, i_q)$ be a maximal path in L such that for any link $i_k i_{k+1}$, $k \in \{1, 2, \ldots, q-1\}$, on this path, $\alpha_{K_{(i_{k+1}, i_k)}}(L)$ and $\alpha_{K_{(i_k, i_{k+1})}}(L)$ in $]0, 1[$. Since the path is maximal $\alpha_{K_{(i_1, j)}} \in \{0, 1\}$ for each $j \in L_{i_1} \backslash \{i_2\}$ and $\alpha_{K_{(i_q, j)}} \in \{0, 1\}$ for each $j \in L_{i_q} \backslash \{i_{q-1}\}$. Let $\varepsilon \in \mathbb{R}$. For $|\varepsilon|$ sufficiently small, define the system of weights α^{ε} which differs from α only on L, and where

– for each link $i_k i_{k+1}$, $k \in \{1, 2, \ldots, q-1\}$, on the path,

$$\alpha^{\varepsilon}_{K_{(i_{k+1}, i_k)}}(L) = \alpha_{K_{(i_{k+1}, i_k)}}(L) + \varepsilon \text{ and } \alpha^{\varepsilon}_{K_{(i_k, i_{k+1})}}(L) = \alpha_{K_{(i_k, i_{k+1})}}(L) - \varepsilon;$$

– for any other link $ij \in L$, $\alpha^{\varepsilon}_{K_{(j,i)}}(L) = \alpha_{K_{(j,i)}}(L)$, and so $\alpha^{\varepsilon}_{K_{(i,j)}}(L) =$

$\alpha_{K_{(i,j)}}(L)$.

As $g^{\alpha}(v^{\Delta_L})$ is a Harsanyi vector payoff, $\alpha(L)$ satisfies (5.1) since assertion 1 is equivalent to assertion 2, and the sharing system p is constructed as in (5.2). Thus, for $|\varepsilon|$ sufficiently small, $g^{\alpha^{\varepsilon}}(v^{\Delta_L})$ is also a Harsanyi payoff vector since

$$\sum_{j \in L_{i_1}} \alpha^{\varepsilon}_{K_{(j,i_1)}}(L) = \sum_{j \in L_{i_1}} \alpha_{K_{(j,i_1)}}(L) + \varepsilon \leq 1,$$

$$\sum_{j \in L_{i_q}} \alpha^{\varepsilon}_{K_{(j,i_q)}}(L) = \sum_{j \in L_{i_q}} \alpha_{K_{(j,i_q)}}(L) - \varepsilon \leq 1$$

and for each other link $ij \in L$,

$$\sum_{j \in L_i} \alpha^{\varepsilon}_{K_{(j,i)}}(L) = \sum_{j \in L_i} \alpha_{K_{(j,i)}}(L) \leq 1.$$

It follows that the convex combination $\big(g^{\alpha^{\varepsilon}}(v^{\Delta_L}) + g^{\alpha^{-\varepsilon}}(v^{\Delta_L})\big)/2$ is also a Harsanyi payoff vector for $|\varepsilon|$ sufficiently small, which contradicts the initial assumption. Therefore, the marginal contribution vectors constitute the profile of the Harsanyi payoff vectors of the form $g^{\alpha}(v^{\Delta_L})$, $\alpha \in \mathcal{A}$. $\square$

From Theorem 3.1 and Theorem 5.1, we derive a new characterization of the Random Tree solutions on $\mathcal{F}$. Denote by $\mathcal{B} \subseteq \mathcal{A}$ the subset of systems of weights α which satisfy (5.1) for each L.

Theorem 5.2 *An allocation rule on $\mathcal{F}$ is a Random Tree solution if and only if it satisfies component efficiency and α-component fairness for some $\alpha \in \mathcal{B}$.*

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